longer than for case (ii) or for constant heat flux at the boundary.

SECOND PROBLEM: ARBITRARY WALL HEAT FLUX

The solution for an arbitrary wall heat flux distribution

$$q = \bar{q}(X)[1 + \sum_{m=1}^{\infty} \{a_m(X)\cos m\theta + b_m(X)\sin m\theta\}]$$

can be obtained in this case by applying Duhamel's superimposition theorem on the solution expressed by equation (9). The final result can be expressed in the form

$$\frac{K(t-t_i)}{a} = 4I_{00}(X) + \sum_{s=1}^{\infty} \beta_{0s}^2 c_{0s} R_{0s}(r^*, \beta_{0s})\, I_{0s}(X)\, e^{-\beta_{0s}^2 X}$$
$$+ \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} \beta_{ms}^2 c_{ms} R_{ms}(r^*, \beta_{ms})\, e^{-\beta_{ms}^2 X}$$
$$\{I_{ms}(X)\cos m\theta + I_{msb}(X)\sin m\theta\} \qquad (12)$$

in which

$$I_{00}(X) = \int_0^X \bar{q}(X)\,dX \qquad I_{0s}(X) = \int_0^X \bar{q}(X)\,e^{\beta_{0s}^2 X}\,dX$$

$$I_{msa}(X) = \int_0^X \bar{q}(X)\,a_m(X)\,e^{\beta_{ms}^2 X}\,dX$$

$$I_{msb}(X) = \int_0^X \bar{q}(X)\,b_m(X)\,e^{\beta_{ms}^2 X}\,dx.$$

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ON THE SOLIDIFICATION OF A WARM LIQUID FLOWING OVER A COLD WALL

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(*Received* 7 *October* 1969 *and in revised form* 12 *December* 1969)

NOMENCLATURE

- a, thermal diffusion coefficient;
- C_p, specific heat;
- h, convective heat-transfer coefficient;
- k, thermal conductivity of solidified material;
- L, latent heat of fusion;
- t, temperature;
- T, dimensionless temperature;
- x, position co-ordinate;
- X, dimensionless co-ordinate;
- δ, thickness of frozen layer;
- δ_∞, thickness of frozen layer at steady state;
- Θ, modified time;
- θ, time;
- ρ, density;
- Bi, Biot number.

Subscripts

- f, at freezing temperature;
- l, liquid phase of solidifying material;
- ∞, steady state;
- w, wall.

INTRODUCTION

Recently there have been some attempts by several authors [1–4] to produce a closed yet simple relation giving the freezing rate of a warm liquid. Most of the solutions available are cumbersome and involve extensive computations. Below, we give a new analytical solution which is in a closed form and easy to use in practical situations.

SOLUTIONS

The system to be analysed is shown in Fig. 1. A liquid at a constant temperature, t_l, flows over a cooler isothermal wall. A frozen layer forms over the wall and reaches a steady state thickness. The question arises as to how the growth of the freezing layer progresses.

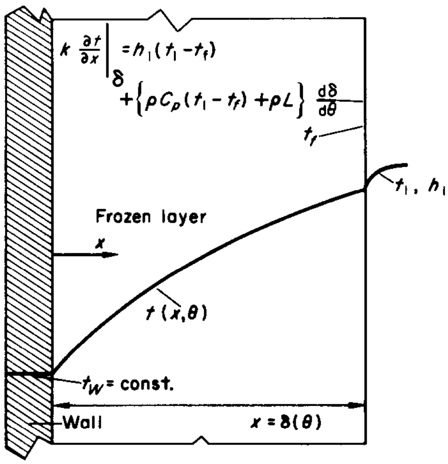

FIG. 1. The sketch of the frozen layer and the temperature distribution within it.

In an attempt to solve the problem it is assumed that the frozen layer properties and the convective heat transfer coefficient remain constant with time.

The set of equations describing the process are:

$$\frac{\partial t}{\partial \theta} = a\frac{\partial^2 t}{\partial x^2}, \quad 0 \leqslant x \leqslant \delta(t), \tag{1}$$

$$x = 0; \quad t(0, \theta) = t_w, \tag{2}$$

$$x \geqslant 0; \quad t(x, 0) = t_l, \tag{3}$$

$$x = \delta; \quad k\frac{\partial t}{\partial x} = h_l(t_l - t_f) + \{\rho C_p(t_l - t_f) + \rho L\}\frac{d\delta}{d\theta}, \tag{4}$$

$$t(\delta, \theta) = t_f \tag{5}$$

$$x \geqslant \delta; \quad t(x, \theta) = t_l > t_f. \tag{6}$$

The solutions of the above equations can be affected by making the following set of transformations:

$$X = x/\delta; \quad \Theta = a\theta; \quad T = t/t_f. \tag{7}$$

Under these transformations equations (1)–(6) reduce to:

$$\delta^2\frac{\partial T}{\partial \Theta} = \frac{\partial^2 T}{\partial X^2} + \frac{X}{2}\cdot\frac{d\delta^2}{d\Theta}\cdot\frac{\partial T}{\partial X}, \tag{8}$$

$$X = 0; \quad T(0, \Theta) = T_w, \tag{9}$$

$$X \geqslant 0; \quad T(X, 0) = T_l,$$

$$X = 1; \quad \frac{\partial T}{\partial X} = \left\{(T_l - 1)Bi + \frac{1}{2}\left[(T_l - 1) + \frac{L}{C_p t_f}\right]\frac{d\delta^2}{d\Theta}\right\}, \tag{10}$$

$$T(1, \Theta) = T_f = 1, \tag{11}$$

$$X \geqslant 1 \; T(X, \Theta) = T_l > 1, \tag{12}$$

where Bi is known as Biot number and equals to $h_l\delta/k$.

Equation (8) can be integrated from X to $X = 1$ to give

$$\delta^2\frac{\partial}{\partial\Theta}\int_1^X T\,d\eta = \left.\frac{\partial T}{\partial X}\right|_X - \left.\frac{\partial T}{\partial X}\right|_1 + \frac{1}{2}\frac{d\delta^2}{d\Theta}\int_1^X \eta\frac{\partial T}{\partial \eta}d\eta$$
$$= \left.\frac{\partial T}{\partial X}\right|_X - \beta + \frac{1}{2}\frac{d\delta^2}{d\Theta}\left\{XT - 1 - \int_1^X T\,d\eta\right\}, \tag{13}$$

where

$$\beta = (T_l - 1)Bi + \frac{1}{2}\left[(T_l - 1) + \frac{L}{C_p t_f}\right]\frac{d\delta^2}{d\Theta}.$$

Using the fact that

$$\int_1^X d\eta \int_1^\eta T\,d\mu = \int_1^X (X - \eta)\,T\,d\eta,$$

then equation (13) can be further simplified to

$$\delta^2\frac{\partial}{\partial\Theta}\int_1^X (X - \eta)\,T\,d\eta = T - 1 - (X - 1)\beta + \frac{1}{2}\frac{d\delta^2}{d\Theta},$$
$$\left\{\int_1^X \eta\,T\,d\eta - (X - 1) - \int_1^X (X - \eta)\,T\,d\eta\right\}, \tag{14}$$

or re-arranging the integrals in the last equation one obtains

$$\delta^2\frac{\partial}{\partial\Theta}\int_1^X (X - \eta)\,T\,d\eta = T - 1 - (X - 1)\beta$$
$$- \tfrac{1}{2}(X - 1)\frac{d\delta^2}{d\Theta} + \frac{1}{2}\frac{d\delta^2}{d\Theta}\int_1^X (2\eta - X)\,T\,d\eta. \tag{15}$$

To make use of the boundary conditions at the wall–frozen liquid interface we put $X = 0$ in equation (15) which results in

$$\frac{d}{d\Theta}\left\{\delta^2\left[\int_0^1 \eta T\,d\eta - \frac{1}{2}\right]\right\} = T_w - 1 + (T_l - 1)Bi + \tfrac{1}{2}$$
$$\times\left[(T_l - 1) + \frac{L}{C_p t_f}\right]\frac{d\delta^2}{d\Theta}, \tag{16}$$

or the re-arrangement of the last equation gives

$$\frac{d}{d\Theta}\left\{\delta^2\left[\int_0^1 \eta\,T\,d\eta - \frac{1}{2}\left(T_l + \frac{L}{C_p t_f}\right)\right]\right\}$$
$$= T_w - 1 + (T_l - 1)Bi. \tag{17}$$

Integrating equation (17) with respect to Θ, finally, one obtains

$$\delta^2 = \frac{\int\limits_0^{\Theta} \left\{(T_w - 1) + (T_l - 1)\frac{h_l\delta}{k}\right\} d\Theta}{\int\limits_0^1 \eta\, T\, d\eta - \frac{1}{2}\left[T_l + \frac{L}{C_p t_f}\right]}. \tag{18}$$

Equation (18) gives an expression for $\delta(t)$, the instantaneous thickness of the frozen layer, directly in terms of the integral of the temperature profile in the frozen layer and the boundary condition at the wall. Since it involves integrals it has the usual advantage of integral formulation in the sense that it is relatively insensitive to small variations in the kernel function [5].

As one does not know the temperature profile in the frozen layer, a different approach has been used, namely, finding the upper and the lower bounds on the integral in the denominator of equation (18).

Since $T_w \leqslant T \leqslant 1$, then for the bounds we find

$$\tfrac{1}{2}T_w \leqslant \int\limits_0^1 \eta\, T\, d\eta \leqslant \tfrac{1}{2}. \tag{19}$$

Using the above inequality in equation (18) one obtains

$$\frac{2}{T_w - [T_l + (L/C_p t_f)]}\int\limits_0^{\Theta}\left\{(T_w - 1) + (T_l - 1)\frac{h_l\delta}{k}\right\} d\Theta \leqslant \delta^2(t)$$

$$\leqslant \frac{2}{1 - [T_l + (L/C_p t_f)]}\int\limits_0^{\Theta}\left\{(T_w - 1) + (T_l - 1)\frac{h_l\delta}{k}\right\} d\Theta, \tag{20}$$

which gives the upper and the lower bounds for $\delta^2(t)$. A better approximation can be obtained by taking the average values of the denominators in the bounds:

$$\delta^2 = \frac{2\int\limits_0^{\Theta}\left\{(T_w - 1) + (T_l - 1)\frac{h_l\delta}{k}\right\} d\Theta}{\frac{1}{2}(T_w + 1) - \left(T_l + \frac{L}{C_p t_f}\right)}. \tag{21}$$

Equation (21) is a general solution giving the growth of the freezing layer. A computer or graphical calculation can be most easily carried out.

However, it is possible to simplify the last relation on physical grounds as follows; if $T_l = 1$ which is another way of saying that the flowing liquid is at the freezing temperature, equation (21) simplifies to

$$\delta = 2\left\{\frac{(t_f - t_w)\,C_p}{(t_f - t_w)\,C_p + 2L}\right\}^{\frac{1}{2}} \sqrt{a\theta}. \tag{22}$$

Also for $t_w \to 0$ the last relation reduces to

$$\delta = 2\left(\frac{C_p t_f}{C_p t_f + 2L}\right)^{\frac{1}{2}} \sqrt{a\theta}. \tag{23}$$

Equation (23) is the same as the equation (19) of [6] when it is assumed that for molten metals and rocks $C_p t_f/2L \sim 1$ and for water $C_p t_f/2L < 1$.

A further interesting conclusion which can be derived from equation (21) is that on differentiating δ^2 with respect to Θ we obtain the steady state thickness of the freezing layer as

$$\delta_\infty = \frac{t_f - t_w}{t_l - t_f} \cdot \frac{k}{h_l}, \tag{24}$$

which is the same as the equation (1) given in [3]. The equation (21) can also be used to calculate the growth of the frozen layer on a wall of finite heat capacity as has been shown elsewhere [5].

CONCLUSIONS

The advantage of the method presented here over the one given in [3] is that the final solutions for the growth of the freezing layer are simple and compact to be of practical use. And furthermore, the solutions are general so that they are also applicable to the cases where the boundary conditions are time dependent (i.e. varying wall temperatures [5]).

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